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FORMULATIONS AND RELAXATIONS FOR A MULTI-ECHELON CAPACITATED LOCATION-DISTRIBUTION PROBLEM

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Abstract

We consider a multi-echelon location-distribution problem arising from an actual application in fast delivery service. We present and compare two formulations for this problem: an arc-based model and a path-based model. We show that the linear programming (LP) relaxation of the path-based model provides a better bound than the LP relaxation of the arc-based model. We also compare the so-called binary relaxations of the models, which are obtained by relaxing the integrality constraints for the general integer variables, but not for the 0-1 variables. We show that the binary relaxations of the two models always provide the same bound, but that the path-based binary relaxation appears preferable from a computational point of view, since it can be reformulated as an equivalent simple plant location problem (SPLP), for which several efficient algorithms exist. We also show that the LP relaxation of this SPLP reformulation provides a better bound than the LP relaxation of the path-based model.
Key Words: multi-echelon location-distribution problem, formulations, relaxations, simple plant location problem.

Résumé

Nous considérons un problème de localisation-distribution à plusieurs échelons provenant d'une application réelle dans le domaine de la livraison par courrier rapide. Nous présentons et comparons deux formulations de ce problème: l'une dans l'espace des arcs, l'autre dans l'espace des chemins. Nous montrons que la relaxation continue du modèle à base de chemins fournit une meilleure borne que la relaxation continue du modèle à base d'arcs. Nous comparons également les relaxations binaires des modèles, qui sont obtenues en relaxant les contraintes d'intégralité associées aux variables entières générales, mais non celles associées aux variables binaires. Nous montrons que les relaxations binaires des deux modèles fournissent la même borne, mais que la relaxation binaire du modèle à base de chemins est préférable puisqu'elle peut être reformulée comme un problème de localisation simple, pour lequel plusieurs algorithmes efficaces existent. Nous montrons aussi que la relaxation continue de cette reformulation fournit une meilleure borne que la relaxation continue du modèle à base de chemins.
Mots-clés: problème de localisation-distribution à plusieurs échelons, formulations, relaxations, problème de localisation simple.

1 Introduction

In this paper, we consider a multi-echelon location-distribution problem arising from an actual application in fast delivery service, first described by Gendron, Semet and Strozyk [2]: a mail-order company offers several products that must be delivered on time to the customers requesting them. To satisfy these requests, the firm operates a multi-echelon distribution system: starting their trips from a small set of hubs (their locations are assumed known and fixed, following a preliminary strategic analysis), a fleet of medium-size trucks delivers the products to depots, where they are transferred on small-size trucks, and then shipped to satellites, where the products are sorted and delivered to the customers. The company exploits existing facilities for the depots and the satellites, but has to pay to use them. The problem is to ensure that customers' requests are satisfied on time at minimum cost, taking into account the transportation costs and the location costs for using the depots and the satellites.

Typically, satellites are neither owned nor rented by the company. They can be warehouses owned by independent carriers or sites such as car parks where items are transferred from one vehicle to another. More traditional distribution systems, which do not use such satellites, can be very costly to run when the demand varies significantly from one period to another. Moreover, when few depots are present, services such as 24 hour-delivery can be cost ineffective or simply cannot be assured when a wide distribution area is considered. The multi-echelon system we consider allows to satisfy time constraints in a cost-effective way. It is also an *adaptive* system, in the sense that satellites can be opened or closed easily according to demand variations.

We model the problem by defining a network for which the only possible connections are those that ensure on time delivery of the products to the customers. In addition, we assume that for each satellite and each product, the set of customers and the routes used to satisfy their requests have been determined in a preprocessing phase. Hence, the model does not include any routing aspect. Each customer in our model represents a set of customers to which the same product is delivered using a single vehicle. The

transportation cost between a satellite and a customer thus corresponds to the cost of the best route determined during this preprocessing phase.

Transportation costs between hubs and depots, and between depots and satellites, vary with the distance travelled, but more importantly, with the number of vehicles used on each arc, each type of vehicle (medium- or small-size truck) having an associated volumetric capacity. A fixed cost is incurred when using any depot, while the satellite location cost increases with the number of batches of products handled at the satellite (a batch corresponds to a fixed number of product units). This cost structure is similar to what can be found in telecommunications network design applications, where multiple facilities, each with an associated capacity, can be installed on the arcs or the nodes. In our problem, vehicles (at the arcs) and product batches (at the satellites) play the role of facilities. Note that this cost structure is more complex than in most location-distribution problems found in the literature, which typically exhibit fixed costs at the nodes and transportation costs that are linear in the number of product units.

Because the locations of the hubs are assumed to be fixed, there are no fixed costs associated to the hubs and we can always assign to each depot its closest hub without losing optimality. This simplification is also performed in the preprocessing phase. Note that we still need to determine how many product units, on how many medium-size vehicles, need to be transported between any depot and its closest hub, but we are now allowed to associate the corresponding decision variables to the depots, instead of the arcs between hubs and depots. The resulting problem can therefore be considered as a *two-echelon* (from depots to satellites, and from satellites to customers) capacitated location-distribution problem.

We present and compare two formulations for this problem: an arc-based model and a path-based model. We show that the linear programming (LP) relaxation of the path-based model provides a better bound than the LP relaxation of the arc-based model. We also compare the so-called *binary relaxations* of the models, which are obtained by relaxing the integrality constraints for all variables, except for the 0-1 design variables that determine which nodes and which arcs should be used to satisfy customers' requests. We

show that the binary relaxations of the two models always provide the same bound, but that the path-based binary relaxation appears preferable from a computational point of view, since it can be reformulated as an equivalent *simple plant location problem* (SPLP), for which several efficient algorithms exist. We also show that the LP relaxation of this SPLP reformulation provides a better bound than the LP relaxation of the path-based model. Finally, we present preliminary computational results on solving a large-scale application with a heuristic method that exploits this SPLP relaxation.

The paper is organized as follows. In Section 2, we introduce the problem notation and a few basic definitions that will be useful when comparing the models. In the next two sections, 3 and 4, we present the arc-based and the path-based models, respectively. In Section 5, we compare the relaxations of the two models. In Section 6, we show that the path-based binary relaxation can be reformulated as an SPLP, and that the LP relaxation of this SPLP reformulation is better than the LP relaxation of the path-based model. All bound relationships between the different models and their relaxations are summarized in Section 7. Section 8 presents some computational results on solving the large-scale application that motivated this study and concludes with the presentation of some avenues for future research.

2 Notation and Definitions

The following sets define the different types of nodes in the network:

D = set of potential sites to locate depots;
S = set of potential sites to locate satellites;
S_i = set of potential sites to locate satellites connected to depot site $i \in D$;
L = set of customers;
L_j = set of customers connected to satellite $j \in S$;
L_i = set of customers connected to depot $i \in D$ from some satellite $j \in S_i$.

The data related to the customer demands and the vehicle capacities are defined as

follows (all values are assumed to be positive):

n_l = number of product units to deliver to customer $l \in L$;
v_l = volume of product units to deliver to customer $l \in L$;
Q = capacity (in number of product units) of one batch of products handled at any satellite;
R_p = volumetric capacity of a small-size vehicle transporting product units to any depot, from its closest hub;
R_s = volumetric capacity of a medium-size vehicle transporting product units from any depot to any satellite.

The location and transportation costs are defined as follows (all values are assumed to be nonnegative):

f_i = fixed cost for using and operating depot $i \in D$;
g_j = cost per Q product units for using and operating satellite $j \in S$;
c_i = transportation cost for using one medium-size vehicle to transport product units to depot $i \in D$ from its closest hub;
c_{ij} = transportation cost for using one small-size vehicle from depot $i \in D$ to satellite $j \in S_i$;
c_{jl} = transportation cost between satellite $j \in S$ and customer $l \in L_j$.

The models we propose are mixed-integer programming (MIP) formulations having three types of variables: continuous, binary and general integer variables. When relaxing all integrality requirements (on binary and general integer variables), we obtain the linear programming (LP) relaxation. For any model M, we will denote its LP relaxation as $LP(M)$. If we relax the integrality constraints on the general integer variables only, we obtain the binary relaxation, which is a mixed 0-1 programming formulation. For any model M, we will denote its binary relaxation as $BIN(M)$.

We will use the following definitions to compare different relaxations for the same minimization problem, where $Z(M)$ denote the optimal objective value of any model M:

Definition 1 *Relaxation M_1 is* **equivalent** *to relaxation M_2 if $Z(M_1) = Z(M_2)$ for all problem instances.*

Note that a model is a relaxation of itself, so we can use the same definition to show that two models are equivalent.

Definition 2 *Relaxation M_1* **dominates** *relaxation M_2 if $Z(M_1) \geq Z(M_2)$ for all problem instances and there exists at least one problem instance such that $Z(M_1) > Z(M_2)$.*

3 Arc-Based Formulation

To derive the arc-based model, the following sets of binary variables are introduced:

$$X_{jl} = \begin{cases} 1, & \text{if some product units are transported between satellite } j \in S \\ & \text{and customer } l \in L_j; \\ 0, & \text{otherwise;} \end{cases}$$

$$W_{ij} = \begin{cases} 1, & \text{if some product units are transported between depot } i \in D \\ & \text{and satellite } j \in S_i; \\ 0, & \text{otherwise;} \end{cases}$$

$$Y_i = \begin{cases} 1, & \text{if some product units are transported to depot } i \in D \\ & \text{from its closest hub;} \\ 0, & \text{otherwise.} \end{cases}$$

In addition, we introduce flow variables representing the volume of product units transported between depots and satellites:

$$V_{ij} = \text{volume of product units transported between depot } i \in D \text{ and satellite } j \in S_i.$$

We also use the following general integer variables to represent the number of batches handled at any satellite and the number of vehicles used on any depot-satellite arc or at any depot:

U_j = number of batches of products handled at satellite $j \in S$;
T_i = number of medium-size vehicles used between depot $i \in D$ and its closest hub;
T_{ij} = number of small-size vehicles used between depot $i \in D$ and satellite $j \in S_i$.

The arc-based formulation of the problem, denoted M_{arc}, can then be written as follows:

$$\min \sum_{i \in D} f_i Y_i + \sum_{j \in S} g_j U_j + \sum_{i \in D} c_i T_i + \sum_{i \in D} \sum_{j \in S_i} c_{ij} T_{ij} + \sum_{j \in S} \sum_{l \in L_j} c_{jl} X_{jl} \tag{1}$$

$$\sum_{j \in S} X_{jl} = 1, \quad \forall l \in L, \tag{2}$$

$$\sum_{i \in D} W_{ij} \leq 1, \quad \forall j \in S, \tag{3}$$

$$\sum_{i \in D} V_{ij} = \sum_{l \in L_j} v_l X_{jl}, \quad \forall j \in S, \tag{4}$$

$$V_{ij} \leq (\sum_{l \in L_j} v_l) W_{ij}, \quad \forall i \in D, \forall j \in S_i, \tag{5}$$

$$\sum_{j \in S_i} V_{ij} \leq (\sum_{l \in L_i} v_l) Y_i, \quad \forall i \in D, \tag{6}$$

$$X_{jl} \leq \sum_{i \in D} W_{ij}, \quad \forall j \in S, \forall l \in L_j, \tag{7}$$

$$W_{ij} \leq Y_i, \quad \forall i \in D, \forall j \in S_i, \tag{8}$$

$$\sum_{l \in L_j} n_l X_{jl} \leq QU_j, \quad \forall j \in S, \tag{9}$$

$$\sum_{j \in S_i} V_{ij} \leq R_p T_i, \quad \forall i \in D, \tag{10}$$

$$V_{ij} \leq R_s T_{ij}, \quad \forall i \in D, \forall j \in S_i, \tag{11}$$

$$U_j \geq 0 \text{ and integer}, \quad \forall j \in S, \tag{12}$$

$$T_i \geq 0 \text{ and integer}, \quad \forall i \in D, \tag{13}$$

$$T_{ij} \geq 0 \text{ and integer}, \quad \forall i \in D, \forall j \in S_i, \tag{14}$$

$$X_{jl} \in \{0, 1\}, \quad \forall j \in S, \forall l \in L_j, \tag{15}$$

$$W_{ij} \in \{0, 1\}, \quad \forall i \in D, \forall j \in S_i, \tag{16}$$

$$Y_i \in \{0, 1\}, \quad \forall i \in D, \tag{17}$$

$$V_{ij} \geq 0, \quad \forall i \in D, \forall j \in S_i. \tag{18}$$

The objective function, (1), consists in minimising all costs incurred by using and operating depots and satellites, as well as transportation costs between hubs and depots, between depots and satellites, and between satellites and customers. Constraints (2) ensure that each customer is being served by a single satellite. Constraints (3) ensure that any satellite, when it is used, is connected to a single depot. Constraints (4) are flow conservation equations at each satellite. The forcing constraints (5) and (6) link together the flow variables and the binary variables: they ensure that no flow can circulate on a network element (depot-satellite arc or depot) that is not used to transport product units. Constraints (7) and (8) are also forcing constraints that link together the different

types of binary variables. Constraints (7) ensure that any customer cannot be routed from a satellite that is not connected to some depot. Similarly, constraints (8) ensure that any satellite cannot be connected to a depot that is not used to transport product units. Constraints (9) ensure that the number of product units handled at a satellite cannot exceed the capacity of product batches. Constraints (10) and (11) ensure that the total volume of all product units transported on a network element (depot or depot-satellite arc) cannot exceed the capacity of the vehicles used on that network element. Other constraints specify the nature of the different types of variables.

Note that the forcing constraints (7) and (8) are redundant. However, it can be beneficial to keep them in the model, since they might improve the lower bounds obtained from the LP relaxation, as stated and proved in the following proposition.

Proposition 3 *Constraints (7) and (8) are redundant in model M_{arc}, but not in its LP relaxation, $LP(M_{arc})$.*

Proof: To show that (7) are redundant, consider an optimal solution to the model without (7) and let us assume that we have, for some $j \in S$, $\sum_{i \in D} W_{ij} = 0$ (clearly, when $\sum_{i \in D} W_{ij} = 1$, the constraints are automatically satisfied). By (5), we have $\sum_{i \in D} V_{ij} = 0$ and then, by (4), we conclude that $X_{jl} = 0$, for all $l \in L_j$. Hence, constraints (7) are redundant.

To show that (8) are redundant, consider an optimal solution to the model without (8) and let us assume that we have, for some $i \in D$, $Y_i = 0$ (if $Y_i = 1$, the constraints are automatically satisfied). By (6), we have $V_{ij} = 0$, for all $j \in S_i$. If $W_{ij} = 1$ for some $j \in S_i$ (which is possible without violating any constraints), we can set $W_{ij} = 0$, for all $j \in S_i$, and obtain another feasible solution having the same objective value, since there are no costs associated to the W variables. Constraints (8) are satisfied by this solution, and therefore redundant in model M_{arc}.

The example illustrated in Figure 1 shows that constraints (7) and (8) are not redundant in $LP(M_{arc})$. This problem instance has two depots, i and i', two satellites, j and j', and two customers, l and l'. We assume that all parameter values (volume and number

of product units requested by the customer, capacities and costs) are equal to 1, except $c_{jl'}$ which is equal to 3. There are two feasible solutions to model M_{arc}:

(1) Send one product unit on path (i, j, l) and one product unit on path (i, j, l'), with an objective value equal to $f_i + 2 \times g_j + 2 \times c_i + 2 \times c_{ij} + c_{jl} + c_{jl'} = 11$;

(2) Send one product unit on path (i, j, l) and one product unit on path (i', j', l'), with an objective function value equal to $f_i + f_{i'} + g_j + g_{j'} + c_i + c_{i'} + c_{ij} + c_{i'j'} + c_{jl} + c_{j'l'} = 10$, so this is the optimal solution.

An optimal solution to $LP(M_{arc})$ follows the same paths to transport product units as in this last solution. In addition, we have $X_{jl} = 1$ and, by (7), $W_{ij} = 1$. Finally, because of constraints (8), we conclude that $Y_i = 1$. Hence, the optimal solution to $LP(M_{arc})$ is the same as that of M_{arc} and $Z(LP(M_{arc})) = 10$. If we remove constraints (8) from $LP(M_{arc})$, we obtain a different conclusion: because $V_{ij} = 1$ and $\sum_{l \in L_i} v_l = 2$, we have $Y_i = 1/2$, by (6). Similarly, if we keep constraints (8), but we remove (7) from $LP(M_{arc})$, we obtain $W_{ij} = 1/2$ by (5), since $V_{ij} = 1$ and $\sum_{l \in L_j} v_l = 2$. In this case, we also have $Y_i = 1/2$. Therefore, if we remove (7) or (8) from the LP relaxation, the optimal objective value is equal to $9\frac{1}{2} < Z(LP(M_{arc}))$. ■

In the last example, the solution to $LP(M_{arc})$ is the same as that of M_{arc}. This is not always the case, as illustrated by the problem instance of Figure 2, which has two depots, i and i', three satellites, j, j' and j'', and three customers, l, l' and l''. We assume that all parameter values (volume and number of product units requested by the customer, capacities and costs) are equal to 1, except $c_{i'j''}$ which is equal to $\frac{5}{8}$. Note that, if we disregard the fixed charges at the depots, the cost of sending one product on any path originating at i and i' is equal to 4 and $3\frac{5}{8}$, respectively. An optimal solution to this problem consists in sending one product unit on each of the three paths (i, j, l), (i, j', l') and (i, j'', l''), with an objective value equal to 13. An optimal solution to $LP(M_{arc})$ splits the flow evenly on the four paths connecting i to l and l', and sends one product unit on the path from i' to l'': $V_{ij} = V_{ij''} = 1/2$ and $V_{ij'} = V_{i'j''} = 1$. As a result, all W variables assume value $1/2$, and we conclude that $Y_i = Y_{i'} = 1/2$. Hence, the optimal objective value of $LP(M_{arc})$ is equal to $\frac{1}{2} \times f_i + \frac{1}{2} \times f_{i'} + 4 \times (\frac{1}{2} \times 4) + 3\frac{5}{8} = 12\frac{5}{8} < Z(M_{arc})$.

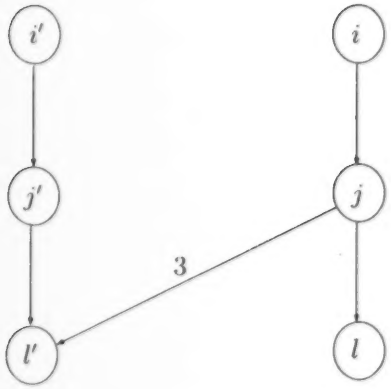

Figure 1: Example showing that constraints (7) and (8) are not redundant in $LP(M_{arc})$

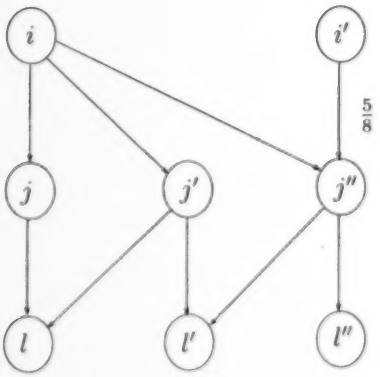

Figure 2: Example showing that $LP(M_{arc})$ can have an optimal fractional solution

The reason for obtaining such fractional solutions in $LP(M_{arc})$ lies in the weakness of the forcing constraints (7), which are defined by using sums of binary variables, instead of individual binary variables. The path-based formulation, which we present next, is an equivalent MIP model that provides a better LP relaxation by introducing tighter forcing constraints.

4 Path-Based Formulation

To define the path-based model, the following set of binary variables is introduced:

$$X_{ijl} = \begin{cases} 1, & \text{if some product units are transported on path } (i,j,l), \\ & i \in D,\ j \in S_i,\ l \in L_j; \\ 0, & \text{otherwise.} \end{cases}$$

These variables are linked to the variables of the arc-based model by the following equations:

$$X_{jl} = \sum_{i \in D} X_{ijl}, \quad \forall j \in S, \forall l \in L_j, \tag{19}$$

$$V_{ij} = \sum_{l \in L_j} v_l X_{ijl}, \quad \forall i \in D, \forall j \in S_i. \tag{20}$$

We can project out of model M_{arc} the variables defined by these equations and remove the flow conservation equations (4), which become redundant. In addition, we replace the weak forcing constraints (7) by stronger ones which make use of tighter links between the binary variables. We then obtain the path-based formulation, denoted M_{path}:

$$\min \sum_{i \in D} f_i Y_i + \sum_{j \in S} g_j U_j + \sum_{i \in D} c_i T_i + \sum_{i \in D} \sum_{j \in S_i} c_{ij} T_{ij} + \sum_{i \in D} \sum_{j \in S_i} \sum_{l \in L_j} c_{jl} X_{ijl} \tag{21}$$

$$\sum_{i \in D} \sum_{j \in S_i} X_{ijl} = 1, \quad \forall l \in L, \tag{22}$$

$$\sum_{i \in D} W_{ij} \leq 1, \quad \forall j \in S, \tag{23}$$

$$\sum_{j \in S_i} \sum_{l \in L_j} v_l X_{ijl} \leq (\sum_{l \in L_i} v_l) Y_i, \quad \forall i \in D, \tag{24}$$

$$X_{ijl} \leq W_{ij}, \quad \forall i \in D, \forall j \in S, \forall l \in L_j, \tag{25}$$

$$W_{ij} \leq Y_i, \quad \forall i \in D, \forall j \in S_i, \tag{26}$$

$$\sum_{i \in D} \sum_{l \in L_j} n_l X_{ijl} \leq QU_j, \quad \forall j \in S, \tag{27}$$

$$\sum_{j \in S_i} \sum_{l \in L_j} v_l X_{ijl} \leq R_p T_i, \quad \forall i \in D, \tag{28}$$

$$\sum_{l \in L_j} v_l X_{ijl} \leq R_s T_{ij}, \quad \forall i \in D, \forall j \in S_i, \tag{29}$$

$$U_j \geq 0 \text{ and integer}, \quad \forall j \in S, \tag{30}$$

$$T_i \geq 0 \text{ and integer}, \quad \forall i \in D, \tag{31}$$

$$T_{ij} \geq 0 \text{ and integer}, \quad \forall i \in D, \forall j \in S_i, \tag{32}$$

$$X_{ijl} \in \{0,1\}, \quad \forall i \in D, \forall j \in S_i, \forall l \in L_j, \tag{33}$$

$$W_{ij} \in \{0,1\}, \quad \forall i \in D, \forall j \in S_i, \tag{34}$$

$$Y_i \in \{0,1\}, \quad \forall i \in D. \tag{35}$$

Proposition 4 *M_{arc} and M_{path} are equivalent.*

Proof:

(1) $Z(M_{arc}) \geq Z(M_{path})$

Consider an optimal solution to M_{arc} and define

$$X_{ijl} = \begin{cases} 1, & \text{if } X_{jl} = 1 \text{ and } W_{ij} = 1, \\ & i \in D,\ j \in S_i,\ l \in L_j; \\ 0, & \text{otherwise.} \end{cases} \tag{36}$$

An equivalent definition is $X_{ijl} = W_{ij} X_{jl}$, $i \in D$, $j \in S_i$, $l \in L_j$, which implies the two following equations:

(a) $\sum_{i \in D} X_{ijl} = (\sum_{i \in D} W_{ij}) X_{jl} = X_{jl}$, $j \in S$, $l \in L_j$ (the last equation follows immediately from constraints (3) and (7));

(b) $\sum_{l \in L_j} v_l X_{ijl} = W_{ij} \sum_{l \in L_j} v_l X_{jl} = W_{ij} \sum_{i' \in D} V_{i'j} = V_{ij}$, $i \in D$, $j \in S_i$ (the last equation follows immediately from constraints (3) and (5)).

Now, we can derive a solution to M_{path} by using definition (36), all other variables in M_{path} having the same values as in the optimal solution to M_{arc}; this solution is feasible since:

constraints (23) and (26) are satisfied, since they are the same as (3) and (8), respectively;

definition (36) implies constraints (25);

by using equations (a), we derive constraints (22) and (27) from (2) and (9), respectively;

by using equations (b), we derive constraints (24), (29) and (28) from (6), (11) and (10), respectively.

By equations (a), this solution has the same objective value as the optimal solution to M_{arc}, which shows that $Z(M_{arc}) \geq Z(M_{path})$.

(2) $Z(M_{path}) \geq Z(M_{arc})$

Consider an optimal solution to M_{path} and derive a solution to M_{arc} by using equations (19) and (20), all other variables assuming the same values as in the optimal solution to M_{path}; this solution is feasible since:

constraints (3) and (8) are satisfied, since they are the same as (23) and (26), respectively;

by using equations (19), we derive constraints (2) and (9) from (22) and (27);

by using equations (20), we derive constraints (6), (11) and (10) from (24), (29) and (28);

by combining equations (19) and (20), we obtain (4);

by summing (25) over $i \in D$, we obtain constraints (7) from equations (19);

by multiplying (25) by v_l and by summing them over $l \in L_j$, we obtain constraints (5).

By equations (19), this solution has the same objective value as the optimal solution to M_{path}, which shows that $Z(M_{path}) \geq Z(M_{arc})$. ■

Note that model M_{path} contains no constraints corresponding to the forcing constraints (5) of M_{arc}. As we have just seen in the proof of Proposition 4, the latter are obtained by aggregation of the forcing constraints (25). Constraints (24), however, cannot be obtained by simple aggregation of the other forcing constraints and, although they are redundant in model M_{path}, they are not in its LP relaxation, as shown in the following proposition.

Proposition 5 *Constraints (24) are redundant in M_{path}, but not in $LP(M_{path})$.*

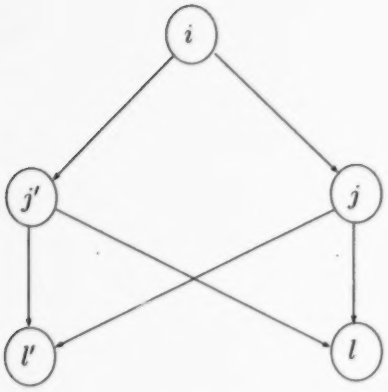

Figure 3: Example showing that constraints (24) are not redundant in $LP(M_{path})$

Proof: To show that (24) are redundant, consider an optimal solution to the model without (24) and assume that we have $Y_i = 0$ for $i \in D$ (when $Y_i = 1$, the constraints are automatically satisfied). By (26), we have $W_{ij} = 0$ for all $j \in S_i$ and then, by (25), we conclude that $X_{ijl} = 0$, for all $j \in S_i$ and $l \in L_j$, which implies that the left-hand side of (24) is equal to 0. Hence, constraints (24) are redundant.

The problem instance displayed in Figure 3 allows us to show that constraints (24) are not redundant in $LP(M_{path})$. In this example, there are one depot, i, two satellites, j and j', and two customers, l and l'. We assume that all parameter values (volume and number of product units requested by the customer, capacities and costs) are equal to 1. An optimal solution to $LP(M_{path})$ splits the flow evenly on all possible paths: $X_{ijl} = X_{ijl'} = X_{ij'l} = X_{ij'l'} = 1/2$. Because of the forcing constraints (25), we have $W_{ij} = W_{ij'} = 1/2$. Now, if only constraints (26) were present, we would have $Y_i = 1/2$, but because of constraints (24), we have $Y_i = 1$, since $\sum_{l \in L_i} v_l = 2$ and $\sum_{j \in S_i} \sum_{l \in L_j} v_l X_{ijl} = 2$. ■

The situation is the same the other way around: if we keep constraints (24) in the formulation, then constraints (26) are redundant in the model, but not in its LP relaxation.

Proposition 6 *Constraints (26) are redundant in M_{path}, but not in $LP(M_{path})$.*

Proof: Consider an optimal solution to the model without (26) and assume that we have $Y_i = 0$ for some $i \in D$ (when $Y_i = 1$, the constraints are automatically satisfied). By (24), we have $X_{ijl} = 0$ for all $j \in S_i$ and $l \in L_j$. If $W_{ij} = 1$ for some $j \in S_i$ (which is possible without violating any constraints), then we can set $W_{ij} = 0$ for all $j \in S_i$ to obtain another feasible solution with the same objective value, since there are no costs associated to the W variables. Constraints (26) are satisfied by this solution, and we conclude that they are redundant in M_{path}.

To show that these constraints are not redundant in $LP(M_{path})$, consider again the problem instance shown in Figure 1. An optimal solution to $LP(M_{path})$ follows the paths (i, j, l) and (i', j', l'). We also have $X_{ijl} = X_{i'j'l'} = W_{ij} = W_{i'j'} = Y_{i'} = 1$, but $Y_i = 1/2$, since $\sum_{l \in L_i} v_l = 2$ and only one product unit is transported on path (i, j, l). By adding constraints (26) to the LP relaxation, we would have obtained $Y_i = 1$. ■

For the problem instances of Figures 1 and 3, the LP relaxation of M_{path} solves with 0-1 values for all binary variables. Unfortunately, this is not always the case. To show this, consider again the problem instance illustrated in Figure 2. As we already know, $Z(M_{path}) = Z(M_{arc}) = 13$ for this problem instance. An optimal solution to $LP(M_{path})$ splits the flow evenly on the two paths from i to l, and sends one product unit on each of the two paths originating at i': $X_{ijl} = X_{ij'l} = 1/2$ and $X_{i'j''l'} = X_{i'j''l''} = 1$. As a result, we have $W_{ij} = W_{ij'} = Y_i = 1/2$ and $W_{i'j''} = Y_{i'} = 1$. Hence, the optimal objective value of $LP(M_{path})$ is equal to $\frac{1}{2} \times f_i + f_{i'} + 2 \times (\frac{1}{2} \times 4) + 2 \times (3\frac{5}{8}) = 12\frac{3}{4} < Z(M_{path})$.

This example also shows that we can have $Z(LP(M_{path})) > Z(LP(M_{arc}))$. This happens because, from an optimal solution to $LP(M_{arc})$, we cannot derive a feasible solution to $LP(M_{path})$ as in the proof of Proposition 4. Indeed, in an optimal solution to $LP(M_{arc})$ for the problem instance illustrated in Figure 2, path (i, j'', l') is used to transport 1/2 product unit and path (i', j'', l'') is used to transport 1 product unit. If we try to derive a feasible solution to $LP(M_{path})$ from this solution, this would imply $W_{ij''} \geq X_{ij''l'} = 1/2$ and $W_{i'j''} \geq X_{i'j''l''} = 1$, hence $W_{ij''} + W_{i'j''} = 3/2 > 1$ and constraints (23) would be violated. The next section further clarifies the relationships

between the relaxations of the two formulations.

5 Comparison of Relaxations

The introduction of the path-based variables is motivated by the goal of deriving a better LP relaxation than the one obtained from the arc-based formulation. This property is stated and proved in the following proposition.

Proposition 7 *$LP(M_{path})$ dominates $LP(M_{arc})$.*

Proof: To show that $Z(LP(M_{path})) \geq Z(LP(M_{arc}))$, we use exactly the same arguments as in the second part of the proof of Proposition 4, where it was shown that $Z(M_{path}) \geq Z(M_{arc})$.
As we have seen at the end of the previous section, Figure 2 provides a problem instance such that $Z(LP(M_{path})) > Z(LP(M_{arc}))$. ■

An interesting alternative to the LP relaxation is the binary relaxation, which is obtained by relaxing the integrality constraints for the general integer variables only, but not for the binary variables. The resulting formulation is a 0-1 MIP problem, having only continuous and binary variables. We then have the following proposition, the proof of which is omitted, since it follows exactly the same arguments as the proof of Proposition 4:

Proposition 8 *$BIN(M_{arc})$ and $BIN(M_{path})$ are equivalent.*

If we are to use the binary relaxation as a basis for solution algorithms, this proposition implies that we cannot favor formulation M_{path} over M_{arc} based on the quality of the lower bound. So, at first sight, it appears that model $BIN(M_{arc})$ is more promising as it involves much less variables. If we look more closely into the structure of $BIN(M_{arc})$, we first have the following proposition that allows to further reduce the number of variables.

Proposition 9 *There exists an optimal solution to $BIN(M_{arc})$ such that*

$$U_j = \frac{1}{Q} \sum_{l \in L_j} n_l X_{jl}, \quad \forall j \in S, \tag{37}$$

$$T_i = \frac{1}{R_p} \sum_{j \in S_i} V_{ij}, \quad \forall i \in D, \tag{38}$$

$$T_{ij} = \frac{1}{R_s} V_{ij}, \quad \forall i \in D, \forall j \in S_i. \tag{39}$$

Proof: Constraints (9), (10) and (11) define lower bounds on the values of each of the variables U_j, T_i and T_{ij}, respectively. Since these variables do not appear in any other constraints and the costs g_j, c_i and c_{ij} are all nonnegative, these variables (now continuous in model $BIN(M_{arc})$) must assume these lower bounds in some optimal solution. ■

By using this proposition, we can reformulate $BIN(M_{arc})$ as follows:

$$\min \sum_{i \in D} f_i Y_i + \sum_{j \in S} \sum_{l \in L_j} (c_{jl} + n_l \frac{g_j}{Q}) X_{jl} + \sum_{i \in D} \sum_{j \in S_i} (\frac{c_i}{R_p} + \frac{c_{ij}}{R_s}) V_{ij} \tag{40}$$

$$\sum_{j \in S} X_{jl} = 1, \quad \forall l \in L, \tag{41}$$

$$\sum_{i \in D} W_{ij} \leq 1, \quad \forall j \in S, \tag{42}$$

$$\sum_{i \in D} V_{ij} = \sum_{l \in L_j} v_l X_{jl}, \quad \forall j \in S, \tag{43}$$

$$V_{ij} \leq (\sum_{l \in L_j} v_l) W_{ij}, \quad \forall i \in D, \forall j \in S_i, \tag{44}$$

$$\sum_{j \in S_i} V_{ij} \leq (\sum_{l \in L_i} v_l) Y_i, \quad \forall i \in D, \tag{45}$$

$$X_{jl} \leq \sum_{i \in D} W_{ij}, \quad \forall j \in S, \forall l \in L_j, \tag{46}$$

$$W_{ij} \leq Y_i, \quad \forall i \in D, \forall j \in S_i, \tag{47}$$

$$X_{jl} \in \{0,1\}, \quad \forall j \in S, \forall l \in L_j, \tag{48}$$

$$W_{ij} \in \{0,1\}, \quad \forall i \in D, \forall j \in S_i, \tag{49}$$

$$Y_i \in \{0,1\}, \quad \forall i \in D, \tag{50}$$

$$V_{ij} \geq 0, \quad \forall i \in D, \forall j \in S_i. \tag{51}$$

This formulation is a network design problem that involves both continuous flow variables and binary variables on each arc. Solving this model appears to be a difficult task.

A result similar to Proposition 9 for $BIN(M_{arc})$ also holds for $BIN(M_{path})$ (we omit the proof, which is essentially the same as that of Proposition 9):

Proposition 10 *There exists an optimal solution to* $BIN(M_{path})$ *such that*

$$U_j = \frac{1}{Q}\sum_{i\in D}\sum_{l\in L_j} n_l X_{ijl}, \quad \forall j \in S, \tag{52}$$

$$T_i = \frac{1}{R_p}\sum_{j\in S_i}\sum_{l\in L} v_l X_{ijl}, \quad \forall i \in D, \tag{53}$$

$$T_{ij} = \frac{1}{R_s}\sum_{l\in L_j} v_l X_{ijl}, \quad \forall i \in D, \forall j \in S_i. \tag{54}$$

By using this proposition, we can reformulate $BIN(M_{path})$ as follows:

$$\min \sum_{i\in D} f_i Y_i + \sum_{i\in D}\sum_{j\in S_i}\sum_{l\in L} C_{ijl} X_{ijl} \tag{55}$$

$$\sum_{i\in D}\sum_{j\in S_i} X_{ijl} = 1, \quad \forall l \in L, \tag{56}$$

$$\sum_{i\in D} W_{ij} \le 1, \quad \forall j \in S, \tag{57}$$

$$\sum_{j\in S_i}\sum_{l\in L_j} v_l X_{ijl} \le (\sum_{l\in L_i} v_l) Y_i, \quad \forall i \in D, \tag{58}$$

$$X_{ijl} \le W_{ij}, \quad \forall i \in D, \forall j \in S, \forall l \in L_j, \tag{59}$$

$$W_{ij} \le Y_i, \quad \forall i \in D, \forall j \in S_i, \tag{60}$$

$$X_{ijl} \in \{0,1\}, \quad \forall i \in D, \forall j \in S_i, \forall l \in L_j, \tag{61}$$

$$W_{ij} \in \{0,1\}, \quad \forall i \in D, \forall j \in S_i, \tag{62}$$

$$Y_i \in \{0,1\}, \quad \forall i \in D. \tag{63}$$

In the objective function (55), we have

$$C_{ijl} = c_{jl} + n_l \frac{g_j}{Q} + v_l(\frac{c_i}{R_p} + \frac{c_{ij}}{R_s}), \quad \forall i \in D, \forall j \in S_i, \forall l \in L_j. \tag{64}$$

In the next section, we show that this model can be reformulated as a simple plant location problem (SPLP), for which large instances can be solved by several efficient algorithms. Therefore, relaxation $BIN(M_{path})$, although equivalent to $BIN(M_{arc})$, appears to be a superior alternative from a computational point of view.

6 SPLP Reformulation of $BIN(M_{path})$

The transformation of $BIN(M_{path})$ into an equivalent simple plant location problem (SPLP) comes from the observation that, in the reformulation derived at the end of the previous section, there are no costs associated to any depot-satellite arc, but only a fixed cost for each depot and a linear cost for each path. For each depot-customer pair (i,l), we associate its closest satellite (with ties being broken arbitrarily):

$$j(i,l) = \arg\min_{j \in S_i} \{c_{jl} + n_l \frac{g_j}{Q} + v_l \frac{c_{ij}}{R_s}\}, \quad \forall i \in D, l \in L_i. \tag{65}$$

We then have the following proposition:

Proposition 11 *There exists an optimal solution to $BIN(M_{path})$ such that:*
For any pair (i,l), $i \in D$, $l \in L_i$, if $X_{ij'l} = 1$ for some $j' \in S_i$, then $j' = j(i,l)$.

Proof: The proposition statement follows if we can show that, in any optimal solution to $BIN(M_{path})$, we have: for any pair (i,l), $i \in D$, $l \in L_i$, if $X_{ij'l} = 1$ for some $j' \in S_i$, then $c_{j'l} + n_l \frac{g_{j'}}{Q} + v_l \frac{c_{ij'}}{R_s} = \min_{j \in S_i}\{c_{jl} + n_l \frac{g_j}{Q} + v_l \frac{c_{ij}}{R_s}\}$. Assume the contrary: in some optimal solution (W,X,Y), there exist $i' \in D$, $l' \in L_{i'}$ and $j' \in S_i$ such that $X_{i'j'l'} = 1$, but $c_{j'l'} + n_{l'} \frac{g_{j'}}{Q} + v_{l'} \frac{c_{i'j'}}{R_s} > \min_{j \in S_{i'}}\{c_{jl'} + n_{l'} \frac{g_j}{Q} + v_{l'} \frac{c_{i'j}}{R_s}\}$. For notational simplicity, let $j'' = j(i',l')$. The hypothesis implies the following inequality:

$$c_{j'l'} + n_{l'} \frac{g_{j'}}{Q} + v_{l'} \frac{c_{i'j'}}{R_s} > c_{j''l'} + n_{l'} \frac{g_{j''}}{Q} + v_{l'} \frac{c_{i'j''}}{R_s}.$$

We distinguish two cases:

(1) $W_{ij''} = 0, \forall i \in D$

We then define a solution $(\overline{W}, \overline{X}, Y)$ to $BIN(M_{path})$ as follows:

$$\overline{X}_{ijl} = \begin{cases} 0, & \text{if } i = i',\ j = j',\ l = l', \\ 1, & \text{if } i = i',\ j = j'',\ l = l', \\ X_{ijl}, & \text{otherwise;} \end{cases}$$

$$\overline{W}_{ij} = \begin{cases} 0, & \text{if } i = i',\ j = j', \\ 1, & \text{if } i = i',\ j = j'', \\ W_{ij}, & \text{otherwise.} \end{cases}$$

This solution is feasible, since all customers' requests are satisfied using the same paths as in solution (W, X, Y), except for customer l', which is now served through path (i', j'', l') instead of path (i', j', l'). This solution has an objective value, $z(\overline{W}, \overline{X}, Y)$, strictly lower than the optimal objective value $z(W, X, Y)$, since:

$z(W,X,Y) - z(\overline{W},\overline{X},Y) = c_{j'l'} + n_{l'}\frac{g_{j'}}{Q} + v_{l'}(\frac{c_{i'}}{R_p} + \frac{c_{i'j'}}{R_s}) - \{c_{j''l'} + n_{l'}\frac{g_{j''}}{Q} + v_{l'}(\frac{c_{i'}}{R_p} + \frac{c_{i'j''}}{R_s})\} = c_{j'l'} + n_{l'}\frac{g_{j'}}{Q} + v_{l'}\frac{c_{i'j'}}{R_s} - \{c_{j''l'} + n_{l'}\frac{g_{j''}}{Q} + v_{l'}\frac{c_{i'j''}}{R_s}\} > 0$. This is a contradiction.

(2) There exists $i'' \in D$ such that $W_{i''j''} = 1$

This case further separates into two subcases:

(2a) $\frac{c_{i'}}{R_p} + \frac{c_{i'j''}}{R_s} > \frac{c_{i''}}{R_p} + \frac{c_{i''j''}}{R_s}$

We then define a solution $(\overline{W}, \overline{X}, Y)$ to $BIN(M_{path})$ as follows:

$$\overline{X}_{ijl} = \begin{cases} 0, & \text{if } i = i',\ j = j',\ l = l', \\ 1, & \text{if } i = i'',\ j = j'',\ l = l', \\ X_{ijl}, & \text{otherwise;} \end{cases}$$

$$\overline{W}_{ij} = \begin{cases} 0, & \text{if } i = i',\ j = j', \\ W_{ij}, & \text{otherwise.} \end{cases}$$

This solution is feasible, since all customers' requests are satisfied using the same paths as in solution (W, X, Y), except for customer l', which is now served through path (i'', j'', l') instead of path (i', j', l'). This solution has an objective value, $z(\overline{W}, \overline{X}, Y)$, strictly lower than the optimal objective value $z(W, X, Y)$, since:

$z(W,X,Y) - z(\overline{W},\overline{X},Y) = c_{j'l'} + n_{l'}\frac{g_{j'}}{Q} + v_{l'}(\frac{c_{i'}}{R_p} + \frac{c_{i'j'}}{R_s}) - \{c_{j''l'} + n_{l'}\frac{g_{j''}}{Q} + v_{l'}(\frac{c_{i''}}{R_p} + \frac{c_{i''j''}}{R_s})\} > c_{j'l'} + n_{l'}\frac{g_{j'}}{Q} + v_{l'}(\frac{c_{i'}}{R_p} + \frac{c_{i'j'}}{R_s}) - \{c_{j''l'} + n_{l'}\frac{g_{j''}}{Q} + v_{l'}(\frac{c_{i'}}{R_p} + \frac{c_{i'j''}}{R_s})\} = c_{j'l'} + n_{l'}\frac{g_{j'}}{Q} + v_{l'}\frac{c_{i'j'}}{R_s} - \{c_{j''l'} + n_{l'}\frac{g_{j''}}{Q} + v_{l'}\frac{c_{i'j''}}{R_s}\} > 0$. Again, this is a contradiction.

(2b) $\frac{c_{i'}}{R_p} + \frac{c_{i'j''}}{R_s} \leq \frac{c_{i''}}{R_p} + \frac{c_{i''j''}}{R_s}$

In this case, we define the set $L(i'', j'') = \{l \in L_{j''} | X_{i''j''l} = 1\}$, i.e., the set of customers l served through path (i'', j'', l). We then define a solution $(\overline{W}, \overline{X}, Y)$ to $BIN(M_{path})$ as follows:

$$\overline{X}_{ijl} = \begin{cases} 0, & \text{if } (i = i',\ j = j',\ l = l') \text{ or } (i = i'',\ j = j'',\ l \in L(i'', j'')), \\ 1, & \text{if } i = i',\ j = j'',\ l = l' \text{ or } l \in L(i'', j''), \\ X_{ijl}, & \text{otherwise;} \end{cases}$$

$$\overline{W}_{ij} = \begin{cases} 0, & \text{if } (i = i',\ j = j') \text{ or } (i = i'',\ j = \ ''), \\ 1, & \text{if } i = i',\ j = j'', \\ W_{ij}, & \text{otherwise.} \end{cases}$$

This solution is feasible, since all customers' requests are satisfied using the same paths as in solution (W, X, Y), except:

for customer l', which is now served through path (i', j'', l') instead of path (i', j', l');

for customers $l \in L(i'', j'')$, which are now served through (i', j'', l) instead of (i'', j'', l).

This solution has an objective value, $z(\overline{W}, \overline{X}, Y)$, strictly lower than the optimal objective value $z(W, X, Y)$, since:

$z(W,X,Y) - z(\overline{W},\overline{X},Y) = c_{j'l'} + n_{l'}\frac{g_{j'}}{Q} + v_{l'}(\frac{c_{i'}}{R_p} + \frac{c_{i'j'}}{R_s}) + \sum_{l\in L(i'',j'')}\{c_{j''l} + n_l\frac{g_{j''}}{Q} + v_l(\frac{c_{i''}}{R_p} + \frac{c_{i''j''}}{R_s})\} - \{c_{j''l'} + n_{l'}\frac{g_{j''}}{Q} + v_{l'}(\frac{c_{i'}}{R_p} + \frac{c_{i'j''}}{R_s})\} - \sum_{l\in L(i'',j'')}\{c_{j''l} + n_l\frac{g_{j''}}{Q} + v_l(\frac{c_{i'}}{R_p} + \frac{c_{i'j''}}{R_s})\} = c_{j'l'} + n_{l'}\frac{g_{j'}}{Q} + v_{l'}\frac{c_{i'j'}}{R_s} - \{c_{j''l'} + n_{l'}\frac{g_{j''}}{Q} + v_{l'}\frac{c_{i'j''}}{R_s}\} + \sum_{l\in L(i'',j'')} v_l\{(\frac{c_{i''}}{R_p} + \frac{c_{i''j''}}{R_s}) - (\frac{c_{i'}}{R_p} + \frac{c_{i'j''}}{R_s})\} > 0.$

In this case also, we have a contradiction. ■

We use the last proposition to define the following variables:

$$X_{il} = X_{ij(i,l)l}, \quad \forall i \in D, \forall l \in L_i. \tag{66}$$

The SPLP reformulation of the path-based binary relaxation, denoted M_{splp}, is then written as follows:

$$\min \sum_{i\in D} f_i Y_i + \sum_{i\in D}\sum_{l\in L_i} C_{il} X_{il} \tag{67}$$

$$\sum_{i\in D} X_{il} = 1, \quad \forall l \in L, \tag{68}$$

$$X_{il} \leq Y_i, \quad \forall i \in D, \forall l \in L_i, \tag{69}$$

$$X_{il} \in \{0,1\}, \quad \forall i \in D, \forall l \in L_i, \tag{70}$$

$$Y_i \in \{0,1\}, \quad \forall i \in D. \tag{71}$$

In the objective function (67), we have

$$C_{il} = C_{ij(i,l)l}, \quad \forall i \in D, \forall l \in L_i. \tag{72}$$

Proposition 12 *M_{splp} is equivalent to $BIN(M_{path})$.*

Proof:

(1) $Z(BIN(M_{path}) \geq Z(M_{splp})$

Consider an optimal solution to $BIN(M_{path})$ and assume, without loss of generality, that this solution satisfies the conditions of Proposition 11. From definition (66), we have:

(a) $X_{il} = X_{ij(i,l)l} = \sum_{j \in S_i} X_{ijl}$, $i \in D$, $l \in L_i$.

We can then derive a solution to M_{splp}, the Y variables retaining the same values as in the optimal solution to $BIN(M_{path})$; this solution is feasible for M_{splp} since:

by using equations (a), we derive constraints (68) from (56);

constraints (69) are satisfied, as otherwise this would imply that for some pair (i', l'), $i' \in D$, $l' \in L_{i'}$, we would have $X_{i'l'} = X_{i'j(i',l')l'} = 1$ and $Y_{i'} = 0$, but this would contradict constraints (59) and (60).

By equations (a), this solution has the same objective value as the optimal solution to $BIN(M_{path})$, which shows that $Z(BIN(M_{path})) \geq Z(M_{splp})$.

(2) $Z(M_{splp}) \geq Z(BIN(M_{path}))$

Consider an optimal solution to M_{splp} and define

$$X_{ijl} = \begin{cases} X_{il}, & \text{if } j = j(i,l),\ i \in D,\ j \in S_i,\ l \in L_j, \\ 0, & \text{otherwise;} \end{cases}$$

$$W_{ij} = \max_{l \in L_j, j = j(i,l)} \{X_{il}\}, i \in D, j \in S_i.$$

These definitions imply the two following equations:

(a) $X_{il} = X_{ij(i,l)l} = \sum_{j \in S_i} X_{ijl}$, $i \in D$, $l \in L_i$;

(b) $W_{ij} = \max_{l \in L_j} \{X_{ijl}\}$, $i \in D$, $j \in S_i$.

We can then derive a solution to $BIN(M_{path})$, the Y variables retaining the same values as in the optimal solution to (M_{splp}); with the exception of (57), this solution satisfies all constraints of $BIN(M_{path})$:

by using equations (a), we derive constraints (56) from (68);

equations (b) imply constraints (59);

by combining the definition of the W variables with constraints (69), we obtain constraints (60);

constraints (58) are implied by (59) and (60).

By equations (a), this solution has the same objective value as that of the optimal solution to M_{splp}.

It remains to show that constraints (57) are satisfied by the solution we have defined, or that we can always derive from it a solution with the same objective value that satisfies them. Let us assume that constraints (57) are not satisfied by the solution we have defined. This implies that there exists some $j' \in S$ such that there at least two pairs (i', l') and (i'', l'') with $i' \neq i''$, $l' \neq l''$, $j' = j(i', l') = j(i'', l'')$ and $X_{i'l'} = X_{i''l''} = 1$. Without loss of generality, let us assume that $C_{i'l''} \leq C_{i''l''}$. We then define a feasible solution $(\overline{X}, Y)$ as follows:

$$\overline{X}_{il} = \begin{cases} X_{il}, & l \neq l'', i \in D, \\ 1, & l = l'', i = i', \\ 0, & l = l'', i = i''. \end{cases}$$

This solution is feasible to M_{splp}, since only the values of the X variables for the pairs (i', l'') and (i'', l'') have been modified, and:

(1) $\sum_{i \in D} \overline{X}_{il''} = \overline{X}_{i'l''} = 1$;

(2) $\overline{X}_{i'l''} = 1 = Y_{i'}$;

(3) $\overline{X}_{i''l''} = 0 < Y_{i''}$.

This solution has an objective value, $z(\overline{X}, Y)$, no greater than the optimal objective value $z(X, Y)$, since:

$z(X, Y) - z(\overline{X}, Y) = C_{i''l''} X_{i''l''} - C_{i'l''} \overline{X}_{i'l''} = C_{i''l''} - C_{i'l''} \geq 0$. If $C_{i''l''} > C_{i'l''}$, we have a contradiction, which implies that constraints (57) are satisfied. If $C_{i''l''} = C_{i'l''}$, solution $(\overline{X}, Y)$ has the same objective value as that of (X, Y), but has one less depot connected to satellite j'. By repeating the same argument a finite number of times, either we obtain a contradiction, or we derive another solution to M_{splp} having the same objective value and satisfying constraints (57). ■

Although M_{splp} is an attractive reformulation of $BIN(M_{path})$, it is still an NP-hard problem. Hence, we might be interested in assessing its LP relaxation and comparing it to $LP(M_{path})$.

Proposition 13 *$LP(M_{splp})$ dominates $LP(M_{path})$.*

Proof: Consider an optimal solution (X, Y) to $LP(M_{splp})$ and derive a solution to $LP(M_{path})$ in the same way as in the second part of the proof of the previous proposition,

where it was shown that $Z(M_{splp}) \geq Z(BIN(M_{path}))$. Equations (a) and (b) are also satisfied and:

by using equations (a), we derive constraints (56) from (68);

equations (b) imply constraints (59);

by combining the definition of the W variables with constraints (69), we obtain constraints (60);

constraints (58) are satisfied, since we have, for each $i \in D$

$$\begin{array}{rcl} \sum_{j \in S_i} \sum_{l \in L_j} v_l X_{ijl} & = & \sum_{l \in L_j, j=j(i,l)} v_l X_{il} \\ & \leq & \sum_{l \in L_i} v_l X_{il} \\ & \leq & (\sum_{l \in L_i} v_l) \max_{l \in L_i} \{X_{il}\} \\ & = & (\sum_{l \in L_i} v_l) Y_i. \end{array}$$

By equations (a), this solution has the same objective value as that of the optimal solution to M_{splp}.

It remains to show that constraints (57) are satisfied by the solution we have defined, or that we can always derive from it a solution with the same objective value that satisfies them. Let us assume that constraints (57) are not satisfied by the solution we have defined. This implies that there exists some $j' \in S$ such that there at least two pairs (i', l') and (i'', l'') with $i' \neq i''$, $j' = j(i', l') = j(i'', l'')$ and

$$X_{i'l'} + X_{i''l''} > 1. \tag{73}$$

Note that we also have

$$X_{i'l'} + X_{i''l''} - (Y_{i'} - X_{i'l''}) \leq 1, \tag{74}$$

since

$$\begin{array}{rcl} X_{i'l'} + X_{i''l''} - (Y_{i'} - X_{i'l''}) & \leq & Y_{i'} + X_{i''l''} - (Y_{i'} - X_{i'l''}) \\ & = & X_{i'l''} + X_{i''l''} \\ & \leq & 1, \end{array}$$

by constraints (68). Inequalities (73) and (74) imply

$$X_{i'l''} < Y_{i'}. \tag{75}$$

Without loss of generality, let us assume that $C_{i'l''} \leq C_{i''l''}$. We then define a feasible solution $(\overline{X}, Y)$ as follows:

$$\overline{X}_{il} = \begin{cases} X_{il}, & l \neq l'', i \in D, \\ Y_{i'}, & l = l'', i = i', \\ X_{i''l''} - (Y_{i'} - X_{i'l''}), & l = l'', i = i''. \end{cases}$$

This solution is feasible for $LP(M_{splp})$, since only the values of the X variables for the pairs (i', l'') and (i'', l'') have been modified, and:

(1) $\sum_{i \in D} \overline{X}_{il''} = \sum_{i \in D, i \neq i', i''} X_{il''} + Y_{i'} + (X_{i''l''} - (Y_{i'} - X_{i'l''})) = \sum_{i \in D} X_{il''} = 1$;

(2) $\overline{X}_{i'l''} = Y_{i'}$;

(3) $\overline{X}_{i''l''} = X_{i''l''} - (Y_{i'} - X_{i'l''}) < X_{i''l''} \leq Y_{i''}$.

This solution has an objective value, $z(\overline{X}, Y)$, no greater than the optimal objective value $z(X, Y)$ since:

$z(X, Y) - z(\overline{X}, Y) = C_{i'l''}X_{i'l''} + C_{i''l''}X_{i''l''} - C_{i'l''}Y_{i'} - C_{i''l''}\{X_{i''l''} - (Y_{i'} - X_{i'l''})\} = (C_{i''l''} - C_{i'l''})(Y_{i'} - X_{i'l''}) \geq 0$. If $C_{i''l''} > C_{i'l''}$, we obtain $(C_{i''l''} - C_{i'l''})(Y_{i'} - X_{i'l''}) > 0$ by (75), which leads to a contradiction and implies that constraints (57) are satisfied. If $C_{i''l''} = C_{i'l''}$, solution $(\overline{X}, Y)$ has the same objective value as that of (X, Y), but (74) implies that $\overline{X}_{i'l'} + \overline{X}_{i''l''} = X_{i'l'} + X_{i''l''} - (Y_{i'} - X_{i'l''}) \leq 1$. By repeating the same argument a finite number of times, either we obtain a contradiction, or we derive another solution to $LP(M_{splp})$ having the same objective value and satisfying constraints (57).

To show an exemple where $Z(LP(M_{splp})) > Z(LP(M_{path}))$, consider again the problem instance illustrated in Figure 2. When reformulated as an SPLP, this problem instance appears as in Figure 4. In an optimal solution to this problem, only depot i is open, with an objective value equal to $Z(LP(M_{splp})) = 13 > 12\frac{3}{4} = Z(LP(M_{path}))$. ■

In the last example, $LP(M_{splp})$ provides an integral optimal solution. Unfortunately, this is not always the case. Krarup and Pruzan [4] give such an example, due to Gomory, where the LP relaxation of the SPLP formulation provides a fractional solution. For the sake of completeness, we show that we can obtain this example from an instance of our problem. Figure 5 illustrates this problem instance, which has three depots, three satellites and three customers. All parameter values are equal to 1. The SPLP reformulation is shown in Figure 6. An optimal solution to this problem consists in

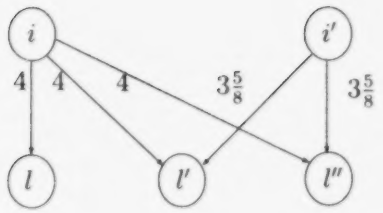

Figure 4: SPLP reformulation of the problem instance of Figure 2

opening any pair of depots, with an objective value equal to 14. The optimal solution to $LP(M_{splp})$ consists in assigning $\frac{1}{2}$ to all variables, with an objective value $Z(LP(M_{splp})) = 13\frac{1}{2} < 14$.

7 Summary of Bound Relationships

The following proposition summarizes the relationships between the different formulations and relaxations introduced so far.

Proposition 14 *The following relationships hold between the optimal values of the different relaxations, where the "$\geq$" sign means a relation of dominance:*

$$\begin{aligned} Z(M_{path}) &= Z(M_{arc}) \\ &\geq Z(BIN(M_{arc})) \\ &= Z(BIN(M_{path})) \\ &= Z(M_{splp}) \\ &\geq Z(LP(M_{splp})) \\ &\geq Z(LP(M_{path})) \\ &\geq Z(LP(M_{arc})). \end{aligned}$$

Proof: The equation on the first line has been shown in Proposition 4.
The inequality on the second line follows immediately by definition of the binary relaxation of M_{arc}. Moreover, the inequality can be strict, since integer solutions to M_{arc} do not necessarily satisfy the equations in the statement of Proposition 9.

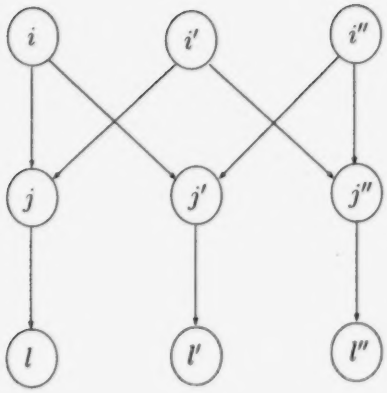

Figure 5: Problem instance showing that M_{splp} can have fractional solutions

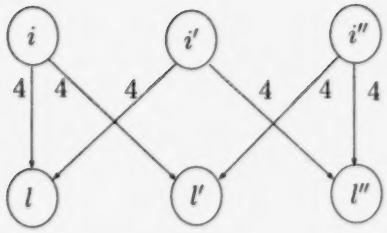

Figure 6: SPLP reformulation of the problem instance of Figure 5

The equation on the third line corresponds to Proposition 8.

The equation on the fourth line has been shown in Proposition 12.

The inequality on the fifth line follows immediately by definition of the LP relaxation of M_{splp}. Moreover, the inequality can be strict, as shown by the problem instance of Figure 5, represented as an SPLP instance in Figure 6.

The inequality on the sixth line has been shown in Proposition 13. The inequality can be strict, as shown by the problem instance of Figure 2.

Finally, the last inequality corresponds to Proposition 7. Again, the problem instance of Figure 2 shows that the inequality can be strict. ■

8 Solving a Large-Scale Application

The actual application that motivated this study has two hubs, 93 depots, 320 satellites, 722 customers, and 458 014 possible connections between the hubs and the customers. Our first attempts at solving this problem were using an arc-based formulation. When running CPLEX (version 7.1) for several days, we obtained a solution with an objective value equal to 1.25×10^8 without being able to prove its optimality. We then focused on developing a more efficient heuristic method, based on the results presented in this paper.

This heuristic method proceeds in three phases. During the initialization phase, a first feasible solution and a lower bound on the optimal value are computed by solving $LP(M_{splp})$ with the dual-ascent procedure and the primal-dual adjustment procedure [6] embedded in DUALOC [1]. This choice was motivated by the fact that the primal-dual adjustment procedure generates several feasible solutions to M_{splp} from which we simply obtain feasible solutions to our problem by rounding up the number of product batches at the satellites and the number of vehicles at the depots and on depot-satellite arcs. This crude heuristic, although not effective (the gap between the initial bounds is 145% for our actual large-scale application), quickly provides an initial feasible solution for the local improvement phase of the heuristic. During this phase, the solution is

gradually improved by proceeding with movements that close either depots or satellites. Each time a candidate movement is considered (using a ranking heuristic that orders the most interesting movements), the impact of closing a node is evaluated by performing DUALOC on the resulting network configuration to identify a new feasible solution. A movement is performed only if it allows to improve upon the current feasible solution. The final phase of the heuristic is a variable neighbourhood descent (VND) [5] that considers two types of neighborhoods. One type of movement consists in connecting a satellite through an alternative depot, while the other neighbourhood is based on connecting a customer to a different satellite. Only improving movements are performed. Our computational results on the actual large-scale problem show a significant improvement in the quality of the feasible solution when the local improvement and VND phases are performed (the gap between the initial lower bound and the best final solution drops to 54%).

These preliminary results are encouraging, but we have made some observations that indicate several sources for improvement and avenues for future research:

- The best solution obtained with our heuristic shows an objective value equal to 1.44×10^8, a gap of 15% compared to the best solution identified by running CPLEX on an arc-based formulation. Moreover, the network configuration indicates that 13 depots and 138 satellites are open in the solution obtained by the heuristic, while the solution obtained with CPLEX shows 25 open depots and 83 open satellites. This clearly indicates that a more elaborate heuristic method is required. In particular, other approaches to obtain initial solutions have to be explored, and other types of neighborhoods need to be visited.

- We have solved M_{splp} exactly with CPLEX and obtained an optimal objective value equal to 9.37×10^7, which also corresponds to the optimal value of $LP(M_{splp})$ (thus, there is no duality gap for this instance, i.e., $Z(LP(M_{splp})) = Z(M_{splp})$). By comparison, when running DUALOC, the best lower bound obtained by the dual-ascent procedure is 9.19×10^7 (a gap of 2%), while the best upper bound obtained

by the primal-dual adjustment procedure is 1.29×10^8 (a gap of 37%). Thus, even though the lower bound obtained by DUALOC is quite good, it is not the case for the upper bound. This suggests that more efficient methods for solving the SPLP relaxation are required (see the recent contribution by Hansen et al. [3], and the references therein, for more sophisticated algorithms for the SPLP).

- Currently, the gap between the best known lower and upper bounds is 33%, which is rather high. In order to improve the bounds, we intend to study the addition of valid inequalities and the development of Lagrangian decomposition methods. The results presented in this paper will be useful with that respect, since the SPLP reformulation of the path-based binary relaxation will allow to solve efficiently the Lagrangian subproblems derived from Lagrangian decomposition schemes.

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References

[1] Erlenkotter, D. (1978). A Dual Procedure for Uncapacitated Facility Location. *Operations Research* 26, 992-1009.

[2] Gendron, B., Semet F., Strozyk, C. (2002). Adaptive Distribution Systems. *TRANSTECH: Transport Technology Product & Process Innovation Management*, 36-42, Presses Universitaires de Valenciennes.

[3] Hansen, P., Brimberg, J., Urosevic, D., Mladenovic, N. (2003). Primal-Dual Variable Neighborhood Search for Bounded Heuristic and Exact Solution of the Simple Plant Location Problem. *Les Cahiers du GERAD* G-2003-64.

[4] Krarup, J., Pruzan, P.M. (1983), The Simple Plant Location Problem: Survey and Synthesis, *European Journal of Operational Research* 12, 36-81.

[5] Mladenovic, N., Hansen, P. (1997). Variable Neighborhood Decomposition Search. *Computers and Operations Research* 24, 1097-1100.

[6] Van Roy, T.J., Erlenkotter, D. (1982). A Dual Procedure for Dynamic Facility Location. *Management Science* 28, 1091-1105.